Reconstruction Properties of Sensing Matrices in Tomographic Particle Imaging Velocimetry

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Abstract. We analyze representative ill-posed scenarios of tomographic PIV with a focus on conditions for unique volume reconstruction. Based on sparse random seedings of a region of interest with small particles, the corresponding systems of linear projection equations are probabilistically analyzed in order to determine (i) the ability of unique reconstruction in terms of the imaging geometry and a single critical sparsity parameter, and (ii) sharpness of the transition to non-unique reconstruction with ghost particles when choosing the sparsity parameter improperly. The sparsity parameter directly relates to the seeding density used for PIV in experimental fluids dynamics that is chosen empirically to date. Our results provide a basic mathematical characterization of the PIV volume reconstruction problem that is an essential prerequisite for any algorithm used to actually compute the reconstruction. Accordingly, we also comment on the role of various reconstruction algorithms currently used in PIV from the optimization point of view. Finally, we outline connections to major developments in other disciplines (compressed sensing) and indicate how the imaging set-up may be further improved. The present paper is a (very) short version of the forthcoming manuscript [8].

1 Introduction

We study the discrete tomography problem in Experimental Fluid Dynamics Tomographic Particle Image Velocimetry (TomoPIV) [1], which received most attention among the different 3D techniques available for measuring velocities of fluids, due to its increased seeding density with respect to other 3D PIV methods. TomoPIV is based on a multiple camera-system, three-dimensional volume illumination and subsequent 3D reconstruction, and employs only few projections due to both limited optical access to wind and water tunnels and cost and complexity of the necessary measurement apparatus. As a consequence, the reconstruction problem becomes severely ill-posed. Ill-posedness is also intimately connected to the particle density, which is a crucial parameter for 3D fluid flow estimation from image measurements. Higher densities ease subsequent flow estimation and increase the resolution and measurement accuracy. On the other side, higher densities also aggravate ill-posedness of the reconstruction problem. A theoretical investigation of this trade-off was lacking so far and is studied in our present work [8]. Below, we merely point out informally few essential points and refer for technical details and a more serious discussion to [8].

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2 Reconstruction

The reconstruction of the 3D image from 2D images employs a standard algebraic reconstruction algorithm: First the problem is discretized, which leads to a linear underdetermined system of equations. Then to cope with ill-posedness, the problem is regularized in terms of a suitable optimization criterium, to select a solution with specific characteristics. Based on the optimization problem a numerical scheme is developed to compute a solution. The classical method from [1] adds to the linear projection equations a positivity constraint and removes ambiguity by entropy maximization. The linear constraints are *not* relaxed. Thus noisy measurements are *not* taken into account. The algorithmic scheme is the classical MART iteration.

Discretization We consider an alternative to the classical voxel discretization [4] based on the assumption that the image I to be reconstructed can be approximated by a linear combination of Gaussian-type *basis functions* $\mathcal{B}_j$, $j = 1, \dots, n$, located at a 3D grid within the volume of interest. The i-th measurement obeys

$$b_i :\approx \int_{L_i} I(z)dz \approx \sum_{j=1}^{n} \int_{L_i} x_j \mathcal{B}_j(z)dz = \sum_{j=1}^{n} x_j \int_{L_i} \mathcal{B}_j(z)dz = \sum_{j=1}^{n} x_j a_{ij} ,$$

where a_{ij} is the value of the i-th pixel if the object to be reconstructed is the j-th basis function. The main task is to estimate the weights x_j corresponding to basis functions from the recorded 2D images, that contain the measurements b_i, and to determine a solution x to $Ax \approx b$.

The matrix A has dimensions (# pixel =: m) $\times$ (# basis functions = n). Since each row indicates those basis functions whose support intersect with the corresponding projection ray, the projection matrix A will be sparse. This property is crucial for examining the limits of the TomoPIV reconstruction problem mathematically.

In the case of a regular grid along with identical basis functions and a parallel ray geometry, see Fig. 1 for an example, the projection matrix can be obtained by a binary geometry matrix, with entry 1 if ray i intersects (neighborhood of) gridpoint j, and 0 otherwise, multiplied by a square and regular band matrix induced by the chosen basis. Without loss of generality, we can further consider just the binary matrix, since the basis matrix does not affect the reconstruction properties of A that is mainly determined by its nullspace.

Regularization The reconstruction of particle volume functions from few projections can be modeled as finding the sparsest solution of the (approximated) underdetermined linear system $Ax \approx b$, since the original particle distribution can be well approximated just by a very small number of active basis functions relative to the number of all possible particle positions in the 3D domain. In general the search for the sparsest solution is intractable (NP-hard), however. Yet, the theory of Compressed Sensing [2,3] shows that one can compute via ℓ_1-minimization the sparsest solution for underdetermined systems of equations *provided* the sensing matrix A satisfies certain conditions that define mathematically ideal sensors. Unfortunately, as shown in [7], all currently available recovery conditions predict an extremely poor performance of the TomoPIV coefficient matrix from this viewpoint, see Fig. 1 and the caption.

3 Sparsity and Improved Reconstruction

In [7] it was shown that if the solution of A is known to be sufficiently sparse and positive it is also the unique positive solution. If A has only nonnegative entries, zero or negligible measurements can be eliminated along with the corresponding incident basis functions. This leads to an "equivalent" feasible set of reduced dimensionality. This procedure is related to multiplicative line-of-sight estimation to fix possible particle positions from [6]. It can be shown that a binary matrix recovers all k-sparse binary vectors if and only if all these reduced systems are overdetermined full-rank systems.

The maximal such k is related to the minimal number of negative (or positive) entries in the sparsest nullspace of A. We estimated the critical k such that for most arbitrary k-sparse vectors the reduced

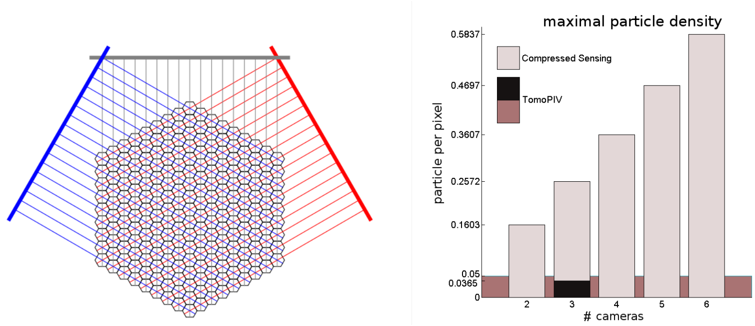

Fig. 1. Left Sketch of a 3-cameras setup in 2D. The corresponding 0/1 sensing matrix A marks in each row corresponding to a projection ray all incident discretization cells by the entry 1, equal to the length of the intersection of the ray with each discretization cell. Matrix $A \in \{0,1\}^{m \times n}$ is underdetermined, with $m = 3(2N+1)$ and $n = 3N^2 + 3N + 1$, where $N + 1$ is the number of cells on each hexagon edge. This geometry can be easily extended to 3D by enhancing both cameras and volume by one dimension, thus representing scenarios of practical relevance as in [5], where a free jet inside an illuminated cylinder was imaged by cameras aligned on a line. Our numerical results cover the range up to $N = 1500$, corresponding to about 6.75×10^6 cells. **Right** Our average case analysis of *correct reconstruction* revealed that TomoPIV matrices perform approximately ten times worse than the mathematically ideal Gaussian ensemble [3], indicated by the dark gray area and light gray bars in the right panel, respectively. The minimal feasible sparsity k (maximal feasible particle density) for a 3-camera system, represented by the black bar, was analytically computed in [7] and depends on the problem size as eqn. (1) predicts.

systems are indeed overdetermined and obtained the relation

$$k(N) \approx 4N^{0.342+0.011\log(N)} \tag{1}$$

depending on the problem size N. Additionally, we proved a tail bound entailing that for increasing large problem sizes $N \to \infty$, the critical k acts like a threshold that sharply discriminates successful reconstruction from failure. Figure 2 illustrates this fact as well as results for sensing matrices that have been improved in a specific way – see [8].

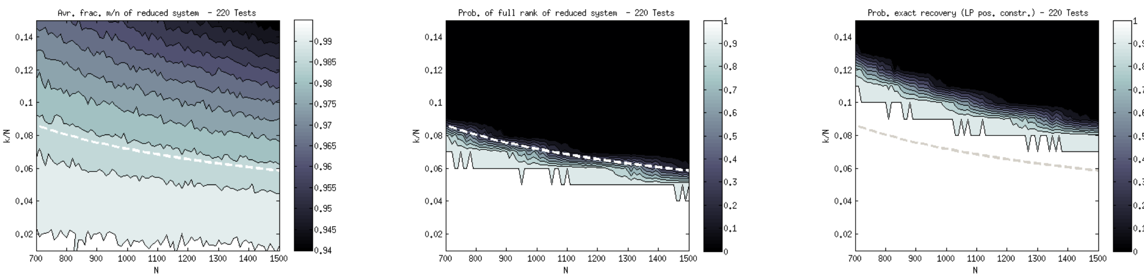

Fig. 2. Left Average undersampling ratio for the reduced systems of equations with the curve from (1). **Middle** The empirical success-failure phase transition for the binary matrix corresponding to the geometry from Fig. 1 **Right** Empirical success-failure phase transition for the improved measurement system along with the phase transition from (1). The results indicate that at least a 150% times better reconstruction performance may be obtained in practice within the considered range of image resolution.

4 Algorithms

In the noiseless case, positivity constraints are sufficient to regularize the reconstruction problem. Therefore one can employ any objective function in order to reconstruct a sparse enough and positive solution. In particular, the non-smooth ℓ_1-regularizer may be replaced by a smooth convex functional, enabling more efficient numerical algorithms. This perspective has not been exploited so far.

On the other hand, when data are noisy as is the case in practice, a suitable distance $D(\cdot,\cdot)$ of Ax to the measurement vector b is tolerated, and we solve the problem

$$\min_x D(Ax, b) \quad \text{subject to} \quad x \geq 0 \,. \tag{2}$$

The current state of the art reconstruction algorithm for TomoPIV in the literature, *Simultaneous Multiplicative Algebraic Reconstruction Technique (SMART)*, minimizes the Kullback-Leibler cross entropy $KL(Ax, y)$ over the nonnegative orthant.

One can also consider nonnegative least squares as done in [9], where the proposed methods outperform SMART in terms of speed.

5 Conclusion

TomoPIV sensing matrices have an extremely poor worst case performance from the viewpoint of compressed sensing, as compared to mathematically ideal sensors. Based on a probabilistical average case analysis, however, we showed an expected performance of the TomoPIV measurement system equal to the low particle densities used by engineers in practice. Furthermore, simulations demonstrate that specific slight random perturbations of the TomoPIV measurement matrix considerably boost the expected reconstruction performance by about 150%. Additionally, sparsity enforcing convex regularization can be replaced by smooth convex optimization, enabling more efficient numerical algorithms. Both properties provide the basis for enhancing future TomoPIV systems.

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