Exercises – Sheet 2

Exercise 1

Prove the following statements:

(a) Let Y and Z be subspaces of the normed space $(X, \|\cdot\|)$ (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every $\alpha \in (0,1)$ there is a $z \in Z$ such that

$$\|z\| = 1, \qquad \|z - y\| \geq \alpha, \quad \forall y \in Y.$$

(b) If a normed space X has the property that the closed unit ball $\mathbb{B}_X = \{x \colon \|x\| \leq 1\}$ is compact, then X is finite dimensional.

Hint: Use part a) and construct a sequence (x_n) of vectors in $\mathbb{B}_X$ with $\|x_n - x_n\| \geq \frac{1}{2} =: \alpha$

Exercise 2

Decide (for each $n \in \mathbb{N}$) whether the open ball $B_1(0) \subset \mathbb{R}^n$ and the closed ball $\mathbb{B}_1(\mathbb{1}) \subset \mathbb{R}^n$ can be separated in sense of the geometric version of the Hahn-Banach theorem (see Thm. 1.1 in lecture notes). If so, can you provide a $x^* \in \mathbb{R}^n$ and a $\alpha \in \mathbb{R}$ such that

$$\langle x^*, y\rangle < \alpha \leq \langle x^*, z\rangle\,, \quad \forall y \in B_1(0), \forall z \in \mathbb{B}_1(\mathbb{1})$$

holds? We denote with $\mathbb{1} = (1, \dots, 1)^\top \in \mathbb{R}^n$ and with $\langle\cdot,\cdot\rangle$ the Euclidean inner product.

Exercise 3

Consider the space of all *step functions*

$$V = \big\{f \colon f(x) = f_i \in \mathbb{Z},\ x \in [i-1, i),\ i \in \{1, \dots, n\}\big\} \tag{1}$$

and the space of all *continuous, piecewise-linear functions*

$$U = \big\{f \in C(\Omega) \colon f' \in V,\ f'(x) = f'_i \in \mathbb{Z},\ x \in [i-1, i),\ i \in \{1, \dots, n\}\big\}, \tag{2}$$

with $f'(x)$ defined by one-sided derivatives at the boundaries of the subintervals. Both spaces U, V are subspaces of

$$X = L^2(\Omega), \quad \Omega = [0, n] \subset \mathbb{R}. \tag{3}$$

We denote the duality pairing of (X^*, X) by $(\cdot,\cdot)$ and the inner product by $\langle\cdot,\cdot\rangle$.

(a) Is U a subspace of V, in particular does $U \subset V$ hold?

(b) Consider the linear functional $v_u^* \in V^*$ that is defined by some fixed function $u \in U$ and is acting on elements of $V \subset X$ by

$$v_u^* \in V^*, \quad v_u^* \colon V \to \mathbb{R}, \qquad f \mapsto v_u^*(f) := (u, f) := \int_\Omega u(x) f(x) dx, \qquad u \in U,\ f \in V. \tag{4}$$

Due to the theorem of Fréchet-Riesz (see lecture notes), there exists a function

$$z = z(v_u^*) \in V \qquad \text{such that} \qquad \langle z, f\rangle = (v_u^*, f)\,, \qquad \forall f \in V. \tag{5}$$

Compute the function z.

Exercise 4

This is the second practical exercise that you are required to solve.

(a) Load image "input1_gray.png" in Python (`imread` from matplotlib.pyplot) from the tutorial webpage. Degrade the image by adding additive zero mean Gaussian noise with standard deviation $\sigma = 0.1$ to each pixel independently (`randn` from numpy.random). Plot the original and noisy image.

(b) A basic approach for denoising an image $f \in \mathbb{R}^{h \times w}$ is local mean filtering (box filtering). The restored intensity at pixel (i, j) is given by

$$u_{ij} = \frac{1}{|\mathcal{N}(i,j)|} \sum_{(k,l) \in \mathcal{N}(i,j)} f_{kl}, \tag{6}$$

which is the average of the grayvalues of the noisy input f within the spatial neighborhood $\mathcal{N}(i, j)$ of the pixel (i, j),. Here we assume a quadratic neighborhood centered arround the pixel (i, j), e.g. 5×5. At boundary pixels the input image is enlarged by zeros accordingly such that the neighborhood size remains constant (zero padding).

Implement a function from scratch with numpy operations, which zero pads the input and calculates the result of the box filter for a given odd filter size.

(c) A criterion to measure the quality of the recovered result with respect to original image is the signal to noise ratio (SNR). The SNR is defined as

$$\text{SNR} = 20 \log \left(\frac{\|f^0\|_2}{\|f^0 - u\|_2} \right), \tag{7}$$

where f^0 is the original clean image (reference) and u is the denoised image.

Calculate and plot the SNR values for the filter sizes $\{3, 5, \ldots, 21\}$. Determine the optimal filter size which is resulting in the cleanest denoised image by selecting the maximum SNR. Finally, plot the denoised image for the optimal filter size. What can you observe in the result?