Variational Approaches to Image Fluid Flow Estimation With Physical Priors

Andrey Vlasenko and Christoph Schnörr

Abstract We present several variational approaches for fluid flow estimation from image sequences in experimental fluid dynamics. These approaches enable the contextual data analysis of particle images based on physical constraints, including bounds on the variation of divergence and vorticity of flow patterns, vanishing divergence for incompressible flows, and iterative estimation-prediction schemes based on vorticity transport for spatiotemporal regularization. All approaches amount to solving convex optimization problems that have unique solutions. They can be computed by standard numerical algorithms exploiting sparsity even for large-scale problems. We also present recent results on the physically consistent denoising of corrupted three-dimensional fluid flow estimates.

1 Introduction

Particle Image Velocimetry (PIV) has been the prevailing image measurement technique for estimating turbulent flows in experimental fluid dynamics for more than two decades [1, 11]. Local flow estimates are obtained by correlating local interrogation windows in subsequent image frames. Window parameters (size, shape) are adapted to local flow variation in order to optimize the trade-off between accuracy and resolution of flow estimation, and noise suppression.

A remarkable fact concerning correlation-based PIV is that prior knowledge, often available in terms of the physics of the underlying problem [16], is not taken into account. Furthermore, estimates at different locations do not explicitly depend on each other, and an overall optimization criterion with respect to all estimates in the whole image domain is lacking. This appears unnatural in view of the interac-

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tion over a large range of scales in turbulent flows. Accordingly, correlation methods bear little resemblance to the Navier-Stokes equations that govern fluid flows.

Methods for combining prior knowledge and data processing have a long history in other fields of image processing. Variational methods, in particular, are amenable to incorporate physical constraints through additional variational terms. In constrast to correlation methods, even the simplest variational method gives rise to algorithms for flow estimation where estimates at different locations explicitly depend on each other. Likewise, the corresponding Euler-Lagrange systems bear some resemblance to the constitutive equations of fluid dynamics. In our opinion, this indicates an important long-term research direction enabling synergy between experimental fluid dynamics and numerical flow simulation.

In the remainder of this paper, we present past and ongoing own work within the DFG priority program 1147[1] on the design of variational methods for fluid flow estimation that incorporate physical prior knowledge [12, 13, 14, 15, 17, 18], along with related work of our group developed in a European project[2] [19, 20, 21, 2]. For related work, we refer to [3, 4, 5, 8, 10] and [7] and references therein.

The focus of our work is on variational methods

- that effectively steer algorithms for image sequence processing towards physically plausible fluid flow estimates, and
- that are mathematically well-posed and have unique solutions which can be computed even for large-scale problems with numerically stable algorithms.

The paper is organized as follows. We discuss unconstrained variational approaches in section 2 and constrained ones in section 3. Recent work on efficient variational techniques for denoising fluid flow estimates both in 2D and 3D is discussed in section 4. We conclude and indicate promising directions for future research in section 5.

Due to lack of space, we only present experimental results for our most recent work. Detailed presentations of all approaches sketched below, including thorough discussions of related work, can be found in the papers referenced above and downloaded from the IPA homepage.

2 Unconstrained Variational Fluid Flow Estimation

Given the intensity function $I\colon \Omega \to \mathbb{R}_+$ (particle image) defined over a 2D or 3D domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, the prototypical variational approach for estimating flow $u\colon \Omega \to \mathbb{R}^d$ for a fixed image frame (point of time) reads

$$\inf_u J(u)\,, \quad J(u) = \int_\Omega \Big\{ (\nabla I \cdot u + \partial_t I)^2 + \lambda\, r(Du, D^2 u) \Big\} dx\,, \quad \lambda > 0\,. \tag{1}$$

[1] http://www.spp1147.tu-berlin.de/

[2] http://fluid.irisa.fr/

Minimizing the functional J entails minimizing the squared residuals of the continuity equation $\frac{d}{dt}I(x,t)=0$ valid for incompressible flows, and a regularizing term $r(\cdot)$ depending on the first- or second-order spatial derivatives of the flow. The latter enforces spatially coherent flow estimates by bounding flow variation depending on a weighting parameter λ.

Basic and advanced examples for regularizers include (formulated for 2D problems $d=2$) [9, 19]

$$r(Du)=\|\nabla u_1\|^2+\|\nabla u_2\|^2\,,\qquad r(D^2u)=\|\nabla \mathrm{div}(u)\|^2+\|\nabla \mathrm{curl}(u)\|^2\,. \tag{2}$$

The former term leads to reasonable estimates of low-turbulent flows [12] whereas the latter provides much more accurate estimates for highly turbulent flows. We point out that this second-order regularizer requires a careful discretization along with an additional term defined on the boundary $\partial\Omega$, in order to obtain unique and stable flow estimates from noisy image data [19]. By choosing a large weight for the term penalizing the divergence, nearly incompressible flows that are typical for 2D scenarios can be conveniently estimated, whereas strictly incompressible flows are better estimated by constrained variational methods as described next.

An extension of the first-order regularization approach to particle *tracking* velocimetry was studied in [13]. Unlike all other approaches discussed in this paper, however, this extension inherently leads to a *non*convex variational problem.

3 Constrained Variational Fluid Flow Estimation

Incompressible flows satisfy the constraint $\mathrm{div}(u)=0$ approximately in 2D settings and strictly so in upcoming 3D scenarios. This section presents two variational approaches for estimating incompressible flows from image sequences.

3.1 Flow Estimation by Flow Control

The basic idea for constrained image flow estimation is to decouple data continuity term and regularization in (1) into an objective function and constraints. This enables regularization by enforcing flow properties strictly. A basic formulation reads [14]

$$\inf_{u,p,f,g} J(u,p,f,g)$$

with

$$J(u,p,f,g)=\int_\Omega \left\{(\nabla I\cdot u+\partial_t I)^2+\lambda\|f\|^2\right\}dx+\gamma\int_{\partial\Omega}\|\partial_{\partial\Omega}g\|^2 ds \tag{3}$$

and subject to

$$\mu \Delta u + \nabla p = f \text{ in } \Omega \,, \quad \operatorname{div}(u) = 0 \text{ in } \Omega \,, \quad u = g \text{ on } \partial\Omega \,. \tag{4}$$

The functional (3) is minimized over all flows satisfying the Stokes equation (4). In comparison to (1), functional (3) additionally includes a multiplier function p related to the incompressibility constraint and control functions f,g as unknowns. These additional degrees of freedom can be determined because the set of admissible flows is constrained, and because control variables steer the constrained flow so as to fit as much as possible the observed optical flow in terms of the time-varying intensity function $I(x,t)$.

Additional regularization and numerical stability is achieved by slightly smoothing the control variables. The constraints reveal $f \propto \Delta u$, i.e. second-order regularization as in (2), but in a physically more strict way.

For scenarios with low Reynold numbers, this method yields physically consistent and accurate flow estimates. In such cases, p and f indeed may be interpreted as pressure and force field, estimated *from image data*. For higher Reynold numbers, these quantities become physically insignificant, yet still ensure highly accurate estimates of turbulent fluid flows through weakly constrained control functions f,g.

3.2 Enforcing Temporal Coherency

A computationally more expensive but still feasible method for additionally enforcing temporal coherency has been suggested in [15]. Flow estimation through constrained variational optimization

$$\inf_u J(u) \,, \quad J(u) = \int_\Omega \Big\{ (\nabla I \cdot u + \partial_t I)^2 + \lambda (\omega - \omega_T)^2 + \kappa \|\nabla \omega\|^2 \Big\} dx \tag{5}$$

subject to the linear constraints

$$\operatorname{div}(u) = 0 \,, \quad \operatorname{curl}(u) = \omega \,, \tag{6}$$

is iterated with flow prediction through the vorticity transport equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega \quad \text{in } \Omega \times [0,T] \,, \quad \omega(x,0) = \omega_0 \,. \tag{7}$$

Each flow estimate u by (5), (6) obtained for some image frame (point of time) defines the initial value $\omega_0 = \operatorname{curl}(u)$ in (7). A prediction ω_T of the flow for the period $[0,T]$ is then computed by (7). This curl field, in turn, is used to regularize the next flow estimate u in (5).

Numerical experiments show that although the implementation of this approach just needs two subsequent frames of an image sequence for estimating u at a specified point of time, the variational estimation-prediction framework effectively encodes a short-time memory over *many* frames that leads to physically consistent flow regularization both in space and time.

4 Constrained Fluid Flow Denoising in 3D

Our current work is focusing on a variational method for denoising fluid flow estimates in a physically consistent way. On the one hand, this task is more involved because we assume to be given as input data just a noisy vector field, *without* having access to the image data from which this fluid flow estimate was computed. On the other hand, our method is widely applicable, because vector fields produced by *any* method can be processed.

A second prominent feature of our method is that no explicit noise model is involved. Rather, the approach relies on modeling the class of physically admissible vector fields and regards anything else as noise. By this a broad range of both random and systematic errors can be removed by the very same approach, including white noise and local bursts of outliers, automatic completion of fluid estimates in local regions whose location is unknown, increasing the spatial resolution of fluid flow estimates, etc. A thorough study of the 2D case is reported in [17, 18]. The method equally applies in 3D, and we report preliminary results for the first time below.

4.1 Variational Approach

The method comprises four steps:

1. **Solenoidal projection.**
   As a first step, the given corrupted vector field d is projected onto the subspace of vector fields with vanishing divergence by solving

$$\Delta q = \mathrm{div}(d)\ , \quad q = 0 \ \text{ on } \partial\Omega\ , \tag{8}$$

   for q and removing the divergence from d,

$$v = d - \nabla q\ .$$

2. **Lowpass filtering.**
   Next we remove high-frequency noise by filtering each component function of v with a Gaussian lowpass filter g_σ,

$$v_g = g_\sigma * v\ . \tag{9}$$

   The cutoff frequency is chosen large enough so as to preserve any relevant signal structure.
3. **Vorticity rectification.**
   The third step of the approach enhances the physical structures of v_g. To this end, we approximate its vorticity field

$$\omega_g = \nabla \times v_g \ , \tag{10}$$

by solving the optimization problem

$$\inf_{\omega} J(\omega) \ , \qquad J(\omega) = \int_{\Omega} \Big\{ \|\omega - \omega_g\|^2 + \alpha \Big(\nu \|\nabla \times \omega\|^2 + 2 \big\langle e(v_g), \omega \big\rangle \Big) \Big\} dx \ , \tag{11}$$

where $e(v_g)$ is a shorthand for the left hand side of the vorticity transport equation

$$e(v) := \partial_t \omega + (v \cdot \nabla)\omega + (\omega \cdot \nabla)v = \nu \Delta \omega \ , \tag{12}$$

whose 3D-formulation differs from the 2D case (7) by an additional term. The criterion (11) embodies a compromise between the approximation of ω_g in (10) and satisfying the vorticity transport equation (12).
Note that $e(\cdot)$ is evaluated in (11) for the vector field v_g computed in the *previous* step. Furthermore, we omitted the time derivative $\partial_t \omega$ in order to restrict the computations to each individual image frame. This turned out to be a reasonable approximation in the cases considered so far. As a result, ω can be computed by just solving a large sparse linear system.

4. **Velocity restoration.**
The final step of our approach recovers an incompressible, denoised vector field from ω by minimizing

$$\inf_{u} J(u) \ , \quad J(u) = \int_{\Omega} \Big\{ \|u - v_g\|^2 + \beta \|\nabla \times u - \omega_g\|^2 \Big\} dx \ , \tag{13}$$

subject to

$$\operatorname{div}(u) = 0 \ . \tag{14}$$

The minimizer u approximates both the velocity fields (9) and the rectified vorticity field (11).
Problem (13), (14) leads to a simple version of the saddle-point problem corresponding to (3), (4). For a consistent discretization with mixed finite elements, we refer to [14].

4.2 Numerical Experiments

We illustrate the performance of our method by visualizing input and output data of two experiments. In both cases, vector fields resulting from a direct numerical simulation [6] served both as ground truth and as input data after corrupting them with white noise.

The first experiment concerns a vertical convection process in 3D with a large noise level. Figure 1 shows, from left to right, the vorticity of the input data, the denoised vector field u from (13), (14), and ground truth. Figure 2 depicts corresponding cross-sections to illustrate the signal-to-noise level.

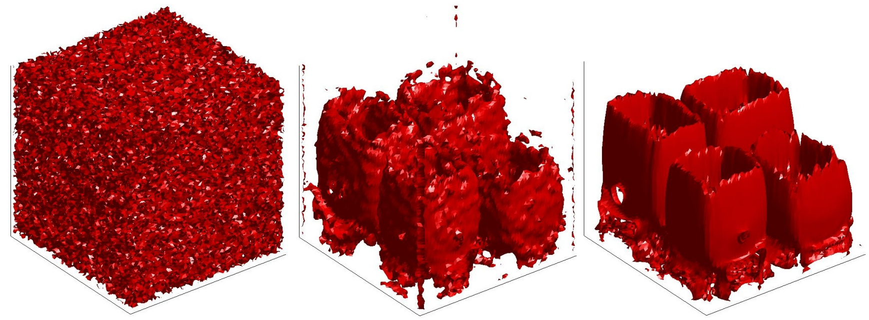

Fig. 1 Instantaneous snapshot of the vorticity of a four-cell vertical convection in three dimensions: noisy input (left), denoised result (center), ground truth (right).

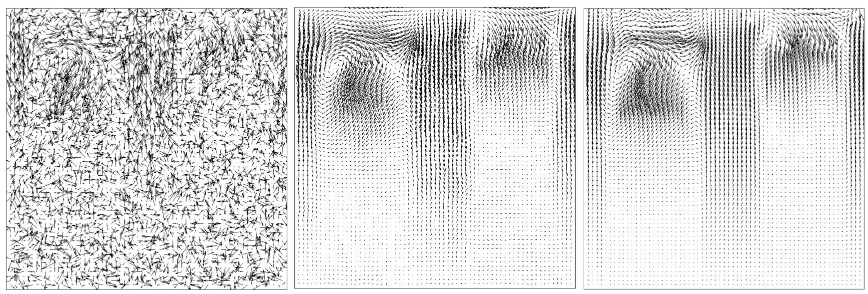

Fig. 2 Vertical cross-sections of the velocity fields through the centers of convective cells. These vector fields correspond to the vorticities shown in Figure 1: noisy input (left), denoised result (center), ground truth (right).

The second experiment concerns a turbulent flow around a cylinder with smaller noise level. Figure 3 shows the vorticity of the input data (left panel) and the denoised output (right panel), respectively, together with two close-up views. Cross-sections analogously to Figure 2 are shown in Figure 4. Close-up views of these vector fields corresponding to the sections shown on the right in Figure 3, are depicted in Figure 5.

5 Conclusion and Further Work

We presented a range of variational methods for physically consistent image processing in experimental fluid dynamics. Our results demonstrate the ability of variational approaches to seamlessly integrate physical prior knowledge. This is particularly relevant for upcoming 3D scenarios in connection with tomographical methods in experimental fluid dyamics.

Another important points is that the mathematical aspects of variational methods and corresponding algorithms are similar to those used in numerical simulations.

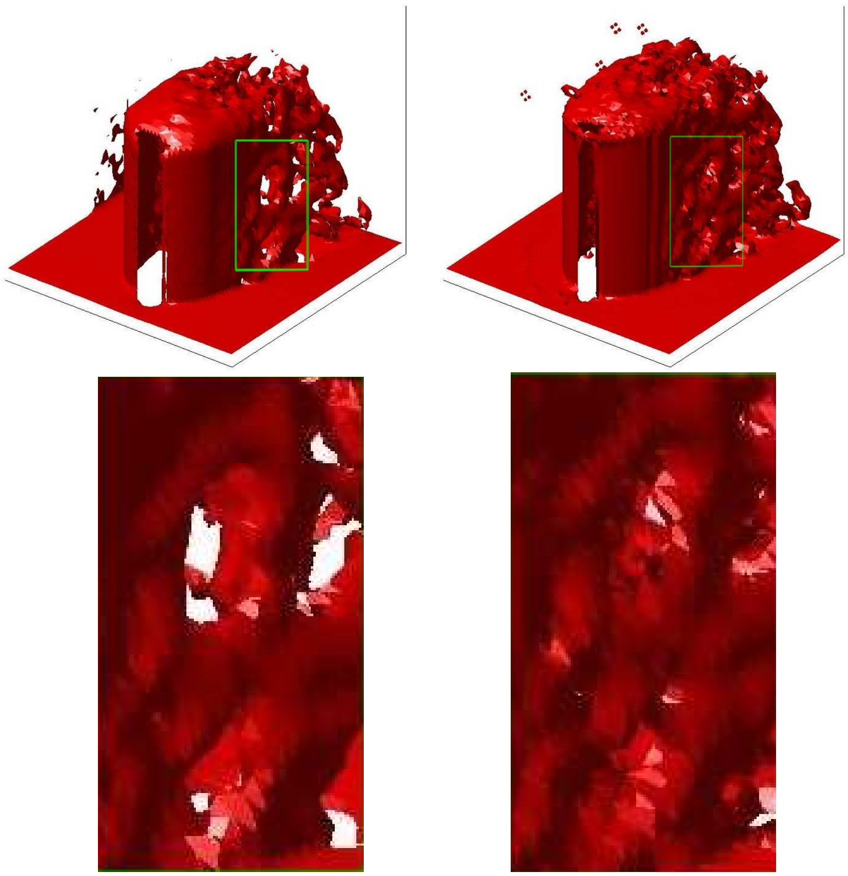

Fig. 3 Top: Instantaneous snapshot of the vorticity of a flow around a cylinder in three dimensions: denoised result (left) and ground truth (right). **Bottom:** the corresponding close-up views.

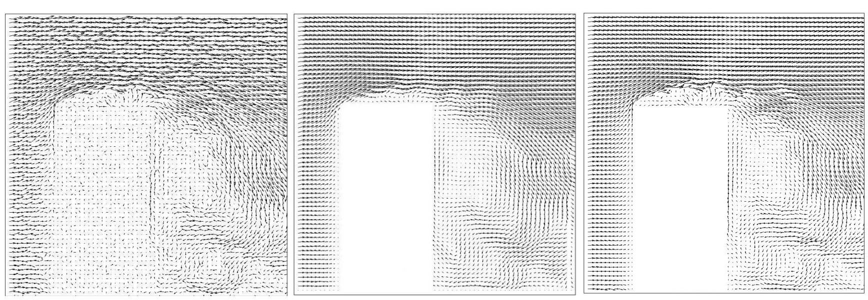

Fig. 4 Vertical cross-sections of the velocity fields through the centers of convective cells. These vector fields correspond to the vorticities shown in Figure 3: noisy input (left), denoised result (center), ground truth (right).

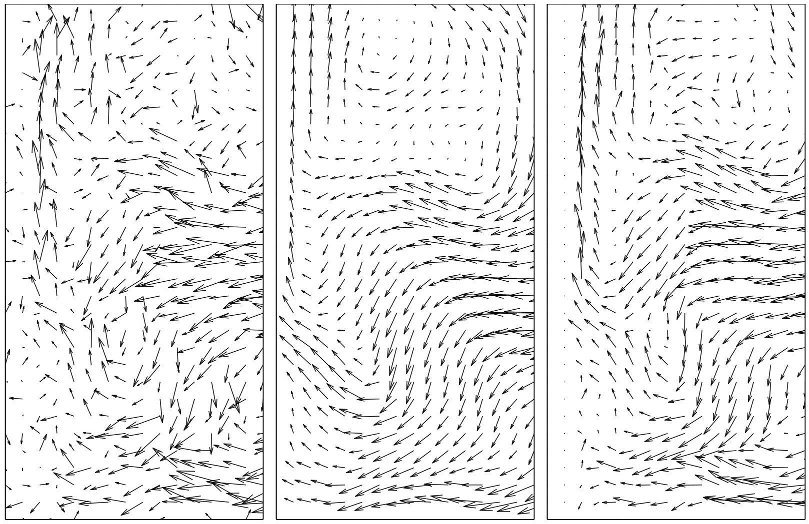

Fig. 5 Close-up views of the vector fields depicted in Figure 4 corresponding to the sections marked in Figure 3: noisy input (left), denoised result (center), ground truth (right).

This may help to tie together in the long run approaches of experimental fluid dynamics and numerical simulation in order to bring to bear the synergy between these complementary fields of research.

This paper mainly focused on aspects of regularization and their physical consistency. The data terms in all approaches above are based on the continuity equation $\frac{d}{dt}I(x,t) = 0$, which is known to be less robust than correlation-based PIV-approaches in very noisy scenarios. Hybrid variational approaches that combine adaptive correlation-based schemes [2] or alternative more advanced local estimation schemes [7] with nonlocal physical priors as presented in this paper (cf. also [8]), is a promising direction for future research.

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