Convex Set-Based Estimation of Image Flows

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Abstract

We introduce a set-based approach for estimating image motion based on an optical flow constraint and a finite number of arbitrary differential constraints describing physically plausible vector fields. Compared to related variational estimation approaches, our approach strictly satifies each separate constraint and becomes not more involved in the presence of higher-order differential operators. The approach is implemented using established subgradient projection schemes onto the set of feasible solutions. Our approach is particularly suited if quantitative prior knowledge about structural flow properties is available, and for the regularized estimation of highly non-rigid image motion.

1. Introduction

Overview and Motivation. Computation of image motion constitutes a very active research field since many years [20, 23]. Recently, there has been an increasing interest in variational methods for the estimation of *highly non-rigid* image flows driven by applications in experimental fluid mechanics, medical imaging, and remote sensing [8, 19, 9, 10, 18]. Typically, such variational approaches comprise some prior regularizing functional (cf., e.g., [23]), and for highly non-rigid image flows there is a need for regularizers involving *higher order partial derivatives* of the flow, so as not to bias motion estimation towards too smooth flow fields.

Further motivation of our work comes from applications of image motion estimation where prior knowledge about structural properties of the image flow to be estimated is available. For example, physical considerations in fluid mechanics may impose an incompressibility constraint $\mathrm{div}(u) = 0$. Likewise, specific camera motions or expected movements of objects in a scene may lead to quantitative constraints on spatial variations of image motion.

Therefore, we investigate in this paper image motion estimation under *several hard* constraints of the form

$$\|L_i u\|^2 \leq \varepsilon_i \ , \quad i = 1, 2, \dots \ , \tag{1}$$

in terms of differential operators L_i involving partial derivatives of the flow $\frac{\partial^\alpha}{x^\alpha} u \ , \ |\alpha| \geq 2$.

Related Work and Contribution. We adopt in this paper a *set-based* framework [4] for the purpose of constrained motion estimation along with the design of corresponding algorithms [2]. Set-based estimation is a well-developed mathematical framework which has found numerous applications in signal processing, estimation, and restoration [4]. In particular, we will draw upon work concerned with the analysis of corresponding algorithms [2, 5, 6], including the intricate but practically important case of inconsistent constraints [7]. Early work employing set-based estimation includes [22, 24].

Concerning the regularizing operators L considered in this paper, *variational approaches* involving $\mathrm{div}(u)$ and $\mathrm{curl}(u)$ include [21, 13, 12, 14, 8], whereas in [10] Δu is used in connection with image registration. Note that using higher order regularization like $L = \Delta u$ necessitates to carefully impose boundary conditions to arrive at a well-posed problem. The correctness of these boundary conditions for a particular image sequence is rarely known in practice, and dropping them naturally leads to *sets* of feasible solutions which, in turn, requires different algorithmic schemes. In the present work, we additionally consider $\nabla \mathrm{div}(u), \nabla \mathrm{curl}(u)$ penalizing only *changes* of these components of the flow gradient, and *arbitrary combinations* of such terms as *hard* constraints.

The scope of this paper is to announce the main ideas underlying our overall approach, due to the limited space. Proofs and further details will be described in a fully elaborated version [25].

2. Discretization

In our work, we use finite-dimensional representations based on the mimetic finite difference method [17, 16].

These representations inherit basic properties and relationships of (continuous) vector analysis. Therefore, they are particularly suited for the estimation and the analysis of highly non-rigid image flows.

We work with 2D regular image grids, square pixels, and side length 1. For both scalar fields and vector fields, two different discrete spaces have to be defined: H_V and H_P, and H_E and H_S, respectively (see [25] for details). These spaces are defined in terms of cell centers $C_{(i+1/2,j+1/2)}$ between four nodes (i,j), $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$, and in terms of side centers $S_{(i,j+1/2)}$, $S_{(i+1/2,j)}$, connecting nodes (i,j), $(i,j+1)$ and (i,j), $(i+1,j)$, respectively.

For example, grayscale images g are represented as discrete cell-centered scalar fields $g \in H_V$, that is gray values $g_{(\alpha,\beta)}$ are defined at cell centers $C_{(i+1/2,j+1/2)}$, $i = 1,\ldots,m-1$, $j = 1,\ldots,n-1$. Velocity fields u, on the other hand, are elements of the discrete vector field in H_S, which means that vectors $u_{S_{(\alpha,\beta)}}$ are located at side centers $S_{(\alpha,\beta)}$. Symbols $USx_{(\alpha,\beta)}$ and $USy_{(\alpha,\beta)}$ are used to denote the x- and y- component of vector u at location (α,β). USx and USy constitute two scalar fields on respective grids. We refer again to [25] for further details.

Discrete operators $\mathbb{D}iv$, $\mathbb{C}\bar{u}rl$, and $\mathbb{G}$ can now be defined as mappings between the spaces H_V, H_P, H_E, H_S. They provide discretizations of their continuous analogues $\operatorname{div} u$, $\operatorname{curl} u$, and ∇ (gradient). Furthermore, they inherit useful properties from vecor analysis, like a discrete but exact version of the Stokes theorem, for example. Moreover, other second-order differential operators, such as $\nabla \operatorname{div} u$ and $\nabla \operatorname{curl} u$, can be defined in this framework without any domain incompatibilities [25]. The inner product in a discrete space S is denoted with angular brackets $\langle\cdot,\cdot\rangle_S$ below.

3. Set-Based Flow Estimation

Let g^1, g^2 denote the vectors of two consecutive frames of an image sequence. Let u denote the flow vectors in H_S as above. Then our discretized version of the optical flow constraint $\nabla g^\top u + \partial_t g = 0$ [15] reads:

$$\mathbb{G}u + g_t = 0 \tag{2}$$

We consider the following sets of feasible vector fields for image motion estimation, defined as convex inequalities and linear equations :

- Bounded gradients:

$$C_1 = \left\{ u \in H_S \ : \ \langle \mathbb{G}USx, \mathbb{G}USx \rangle_{H_E} \le \varepsilon_1 \right\} ,$$
$$C_2 = \left\{ u \in H_S \ : \ \langle \mathbb{G}USy, \mathbb{G}USy \rangle_{H_E} \le \varepsilon_2 \right\}$$

where where $\langle\cdot,\cdot\rangle_{H_E}$ denotes the inner product defined in the discrete vector space H_E.

- Bounded divergence and curl:

$$C_3 = \left\{ u \in H_S \ : \ \langle \mathbb{D}ivu, \mathbb{D}ivu \rangle_{H_V} \le \varepsilon_3 \right\} ,$$
$$C_4 = \left\{ u \in H_S \ : \ \left\langle \mathbb{C}\bar{u}rlu, \mathbb{C}\bar{u}rlu \right\rangle_{\overset{0}{H_P}} \le \varepsilon_4 \right\}$$

where the discrete scalar fields $\mathbb{D}ivu$ and $\mathbb{C}\bar{u}rlu$ are defined in different spatial domains due to u is in predefined in H_S space.

- Bounded gradients of divergence and curl:

$$C_5 = \left\{ u \in H_S \ : \ \left\langle \bar{\mathbb{G}}(\mathbb{D}ivu), \bar{\mathbb{G}}(\mathbb{D}ivu) \right\rangle_{H_S} \le \varepsilon_5 \right\} ,$$
$$C_6 = \left\{ u \in H_S \ : \ \left\langle \mathbb{G}(\mathbb{C}\bar{u}rlu), \mathbb{G}(\mathbb{C}\bar{u}rlu) \right\rangle_{H_S} \le \varepsilon_6 \right\}$$

- Solenoidal and irrotational vector fields:

$$C_7 = \left\{ u \in H_S \ : \ \mathbb{D}ivu = 0 \right\} ,$$
$$C_8 = \left\{ u \in H_S \ : \ \mathbb{C}\bar{u}rlu = 0 \right\}$$

- Finally, the feasible set due to the data constraint (2) is:

$$C_0 = \left\{ u \in H_S \ : \ \langle w, w \rangle_{H_V} \le \varepsilon_0 \right\}$$

We define next our approach to image motion estimation as a *convex feasibility problem (CFP)*. Let $I \subset \{1,\ldots,8\}$ denotes any subset of the sets C_i of feasible vector fields defined above. Then we define as a solution any point $u \in C \subset H_S$ in the convex set:

$$C = \bigcap_{i \in \{0\} \cup I} C_i \ , \quad I \subset \{1,\ldots,8\} \tag{3}$$

The *boundedness* of C, *feasibility* (existence) and *stability*, i.e. the well-posedness, of solutions to (3) are of major interest. For a corresponding analysis, we refer to [25].

4. Algorithms

In connection with signal recovery, there is a vast literature on algorithm design for computing a feasible point in the intersection of convex constraint sets [2, 3]. A broad class thereof leads to relaxed subgradient projection iterative schemes of the form:

$$u^{k+1} = u^k + \lambda \left(\sum_i \omega_i A_i^k u^k - u^k \right) \ , \ \omega_i > 0 \ , \ \sum_i \omega_i = 1$$

Individual schemes differ with respect to the control scheme (in the sequential operating mode) and the specific form of the mappings A_i^k involving projections onto simple convex sets which are locally defined in terms of the subgradient of the distance of the current iterate to the feasible set C.

Due to the limited space, we refer to [25] for a survey and comparison in connection with our approach.

5. Experiments

The objective of this section is to present and discuss preliminary experimental results illustring some specific properties of our approach. We will refer to the constraint sets C_i, $i = 0, \ldots, 8$, defined in section 3. In order to elucidate the principle points and for comparison, we used ground truth data in the first experiment. A more extensive experimental evaluation will be provided in [25].

Error Measures. Since [1], it has become standard practice to evaluate motion estimates by computing the average angular and norm error, repectively, induced by the inner product of the space $((L^2)(\Omega))^2 = L^2(\Omega) \times L^2(\Omega)$. In many applications, however, the structure of motion fields in terms of partial derivatives is equally important. Accordingly, we adopt the average angular and norm error measures but use the inner products of the Sobolev space $(H^1(\Omega))^2$:

$$\langle u, v \rangle_{sob} = \langle u, v \rangle_{(L^2)^2} + \langle \nabla u_1, \nabla v_1 \rangle_{(L^2)^2} + \langle \nabla u_2, \nabla v_2 \rangle_{(L^2)^2} \tag{4}$$

and the space $H(\mathrm{div}; \Omega) \cap H(\mathrm{curl}; \Omega)$ (see, e.g., [11]):

$$\langle u, v \rangle_{dc} = \langle u, v \rangle_{(L^2)^2} + \langle \mathrm{div}\, u, \mathrm{div}\, v \rangle_{L^2} + \langle \mathrm{curl}\, u, \mathrm{curl}\, v \rangle_{L^2} \tag{5}$$

Inappropriate Boundary Conditions. Variational methods for motion estimation by minimizing a corresponding functional [23] impose *natural boundary* conditions on minimizing flows. In case of the prototypical approach of Horn and Schunck [15], for example, these boundary conditions are: $\partial u_1 / \partial n = 0$, $\partial u_2 / \partial n = 0$, where n denotes the outer unit normal with respect to the image domain Ω. Unless specific corresponding prior knowledge is available, these boundary conditions do *not* hold for the *true* flow to be recovered from an image sequence.

To clearly illustrate this point, Figure 1 shows a flow field according to a "zooming" camera (top/left, $|u| < 1.4$). The natural boundary conditions do *not* hold for this motion field. The image top/right was warped using the flow on the left, for the purpose of motion estimation. The (wrong) natural boundary condition lead to *non-locally* distributed error (4) for the variational method (bottom/left) up to $|u_{err}| \leq 0.17 \approx 12\%$. Our set-based approach does not suffer from this drawback (bottom/right; constraints C_1, C_2).

Estimating Laminar Flows. Figure 2 shows an image sequence visualizing the flow behind a landing airplane (made visible by smoke). An important pre-processing step for extracting structural properties of the flow in terms of irrotational and solenoidal flow components (see [8, 9]) is to estimate and to compensate the laminar flow-field component, i.e. $\mathrm{div}(u_{lam}) = \mathrm{curl}(u_{lam}) = 0$. Typically, this is done by applying a variational approach [15] with a large smoothness weight in order to penalize the curl and div component of the vector gradient. Figure 2 shows that this method yields a smooth flow indeed, but not a laminar flow. Figure 2 shows also that our approach using the constraints C_7, C_8 computed a physically plausible result.

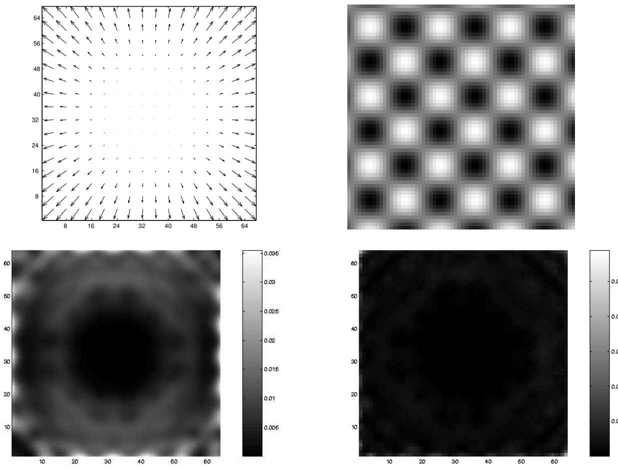

Figure 1. Synthetic image sequence to demonstrate the influence of (typically wrong) natural boundary conditions imposed by variational methods (bottom left) as opposed to our set-based approach (bottom right).

6. Conclusion

We proposed a set-based approach to image motion estimation which allows to incorporate hard constraints with respect to qualitative and quantitative structural flow properties. Unlike variational approaches, higher-order differential constraints do not complicate the computation, and inadequate boundary conditions can be avoided. The approach is particularly suited for estimating physically plausible flows from image sequences of non-rigid fluids.

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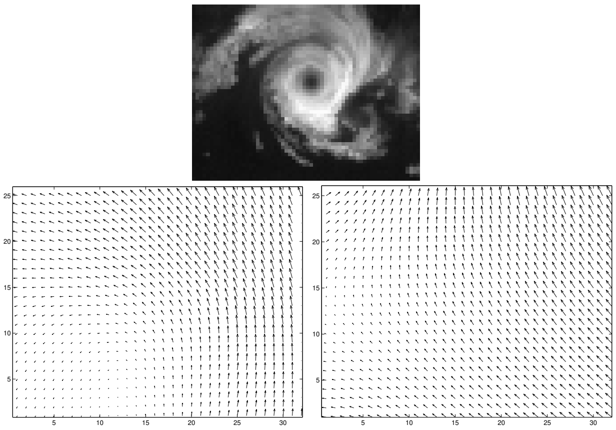

Figure 2. ***Top*** **A fluid image sequence making the flow behind an airplane visible (top). The objective is to estimate the laminar flow-component for subsequent orthogonal flow-decomposition (see [8, 9]).** ***Bottom left:*** **A variational approach using a large regularizing weight yields a smooth flow which is not laminar, however.** ***Bottom right:*** **A truly laminar flow recovered with our set-based approach.**

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