A Linear Programming Approach to Limited Angle 3D Reconstruction from DSA Projections

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Abstract. We investigate the reconstruction of vessels from a small number of Digital Subtraction Angiography (DSA) projections acquired over a limited range of angles. Regularization of this strongly ill-posed problem is achieved by (i) confining the reconstruction to binary vessel / non-vessel decisions, and (ii) by minimizing a global functional involving a smoothness prior. A suitable extension of the standard linear programming relaxation to this difficult combinatorial optimization problem is exploited. Our approach successfully reconstructs a volume of $100 \times 100 \times 100$ voxels including a vascular phantom which was imaged from three projections only.

1 Introduction

Coronary heart diseases and strokes caused by aneurysms and stenosis are world wide the number one killers. For that reason medical imaging that visualizes vascular systems and their 3D structure is of highest importance for many medical applications.

The process of computing the 3D density distribution within the human body from multiple X-ray projections is well understood. Today filtered back-projection is the fundamental algorithm for Computerized Tomography. This algorithm, however, has its limitations. A necessary condition for its success is the rotation of the X-ray tube of at least 180 degrees plus fan angle and the acquisition of a large number of projections. There are prospective applications of 3D where the technical setup does not allow for 180 degree rotations and therefore filtered back-projection cannot be applied. For instance, the reconstruction of the coronary vessels of the moving heart using the Feldkamp algorithm requires so many data, impossible to capture by C-arm systems used during interventions. The conclusion is that new algorithms are required to compute 3D data sets out of a limited range of angles and a small number of X-ray images to push the application of 3D imaging.

In this contribution we investigate the reconstruction of vessels from a small number of DSA projections acquired over a limited range of angles. We use DSA images, i.e. vessels are filled with contrast agent. The background is supposed to be homogeneous. Therefore, we make use of the knowledge that the reconstructed function will contain only two values: either vessel or background. This is the basic prerequisite for Discrete Tomography studied in this work. The restriction to a binary function to be reconstructed compensates for the lack of projections that are usually required for density reconstruction.

2 State of the Art

One way to describe the reconstruction problem is a system of linear equations, $Ax = b$. Thereby each column of the matrix A corresponds to a voxel and each row to a projection ray. With other words the matrix entry $a_{i,j}$ represents the contribution of the j-th voxel to the i-th ray. This matrix is usually sparse since each ray traverses only a small subset of voxels.

Considering only the linear equations it is not possible to force the voxels to be either zero or one. Therefore the linear system is embedded into a linear program where the voxels are kept at least within the interval $[0, 1]$.

In the literature on discrete tomography two linear programming approaches are known. The first one (equation 1) suggested by Fishburn, Schwander, Shepp, and Vanderbei [2] optimizes the dummy functional "zero" subject to the linear projection constraints. Thus, any interior point method for solving large scale linear programs can be used for computing some feasible point in the constraint set.

$$(FSSV) \qquad \min_{x \in \mathbb{R}^n} 0^\top x \;, \quad Ax = b \;, \quad 0 \leq x_j \leq 1 \;, \; \forall j \tag{1}$$

The second approach (equation 2) suggested by Gritzmann, de Vries, and Wiegelmann [3] replaces the dummy functional by the inner product of the one-vector with the vector indexing the unknown voxel samples. Furthermore, the linear projection equations are changed to linear inequalities. This "best inner fit" criterion thus aims at computing a maximal volume among all solutions not violating the projection constraints.

$$(BIF) \qquad \max_{x \in \mathbb{R}^n} e^\top x \;, \quad Ax \leq b \;, \quad 0 \leq x_j \leq 1 \;, \; \forall j \tag{2}$$

3 Main Contribution

Presuming a homogeneous dispersion of the contrast agent within the vessels coherent solutions are more realistic. Both approaches mentioned above do not exploit any spatial context. As a result, spatially incoherent and thus less plausible solutions may be favored by the optimization process.

A common remedy is to include smoothness priors into the optimization criterion. As we deal with integer solutions however, this further complicates the combinatorial optimization problem. Furthermore, smoothness priors lead to quadratic functionals which cannot be tackled by linear programming relaxations.

Inspired by recent progress of J.M. Kleinberg and E. Tardos [4] concerning metric labeling problems, we introduce auxiliary variables to represent the absolute deviation of adjacent entities. By this, spatial smoothness can be measured by a linear combination of auxiliary variables, leading to an extended linear programming approach. As we have already shown in [1], this method leads to much better results in case of 2 dimensional data. In this work we extend [1] to the 3D case and study its medical applicability.

4 Approach

4.1 Discretization

The 3D-function to be reconstructed is currently represented by Haar basis functions with 1-voxel support. A discrete representation of the imaging geometry is achieved by non-uniformly sampling the function along the projection direction. For each projection ray, this yields a linear combination of contributions from basis functions intersecting with the ray. Each contribution is given by the line integral over the intersection. Assembling all contributions into a linear system finally represents the imaging process. For more details we refer the reader to [1,5].

4.2 Preprocessing

For the reconstruction of a $100 \times 100 \times 100$ volume a linear system of one million unknowns has to be solved. All voxels touched by rays with zero projection value must necessarily be zero and can therefore be removed. In case of a vascular system it is possible to reduce the amount of unknowns significantly since the vessels (non-zero voxels) take only a small partition of the whole volume. The remaining voxels constituting the so called "peel volume" are determined in the subsequent reconstruction process.

4.3 Reconstruction

The main idea is to rewrite the linear programs, shown in equation 1 and 2, as follows:

$$(FSSV2) \qquad \min_{x \in \mathbb{R}^n} 0^\top x + \frac{\alpha}{2} \sum_{\langle j,k \rangle} |x_j - x_k| \,, \quad Ax = b, \quad 0 \le x_i \le 1, \quad \forall i \quad (3)$$

$$(BIF2) \qquad \min_{x \in \mathbb{R}^n} -e^\top x + \frac{\alpha}{2} \sum_{\langle j,k \rangle} |x_j - x_k| \,, \quad Ax \le b, \quad 0 \le x_i \le 1, \quad \forall i \quad (4)$$

The last term in the objective functions (equations 3 and 4) measures the difference between adjacent voxels (6-neighborhood), denoted by $\langle \cdot, \cdot \rangle$. For details on how to cast the equations 3 and 4 in linear programming relaxations of the underlying combinatorial 0/1-optimization problems we refer to [1,4].

4.4 Postprocessing

The linear programming step results in a solution vector x with each component $0 \leq x_i \leq 1$. In order to obtain a binary solution we simply used a threshold at 0.2 which led to reasonable results in our experiments, but more sophisticated rounding techniques can be used as well (see [4,6]).

5 Results and Discussion

For evaluation purposes we constructed a vascular phantom which was scanned from several directions by a C-arm system. However, only a few projections are actually used for reconstruction. Each image contained 1024×1024 greyvalue pixels.

Figure 1 shows the reconstruction of the phantom from three projections with (BIF) and $(BIF2)$. The result of $(BIF2)$ is smoother than (BIF) which can better be seen in the closeup, shown in figure 2.

Comparing the (BIF) approach of Gritzmann et al. to $(BIF2)$, the latter yields reconstructions that are less spread out over the entire volume. As can be seen, the smoothness prior leads to significant improvement of reconstruction quality. This clearly shows that the linear programming formulation based on auxiliary variables yields a convex relaxation of the combinatorial optimization problem which is tight enough to compute a good local minimum with standard interior point solvers.

The reconstruction with $(FSSV)$ and $(FSSV2)$ was not possible for real data since the problem becomes infeasible due to the strong equality constraints on the matrix A and noisy projection data.

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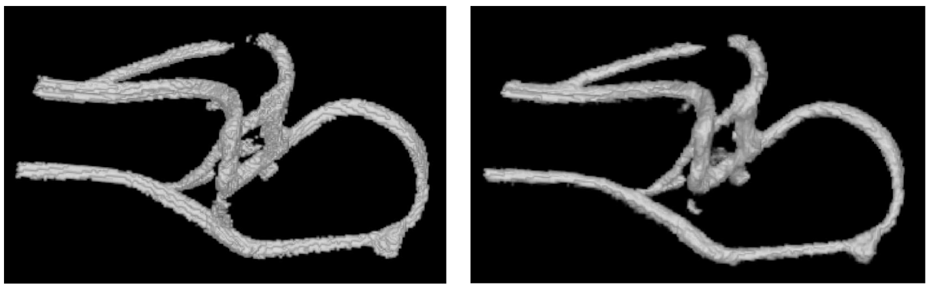
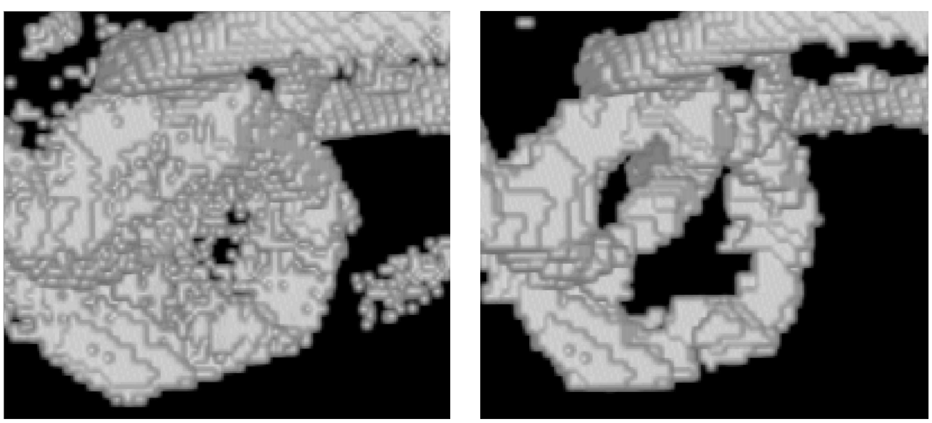


(BIF) $(BIF2)$

Fig. 1. Reconstruction of a $100 \times 100 \times 100$ volume from the vascular phantom using only three projections (0, 50, and 100 degree) with (BIF) and $(BIF2)$.



(BIF) $(BIF2)$

Fig. 2. Closeup of the phantom which was reconstructed with (BIF) and $(BIF2)$.